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LETTER TO THE EDITOR

Approximate Lie and Lie–Bäcklund symmetry of the Kuramoto–Shivashinsky equation

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Abstract. Following an idea of Baikov *et al*, it has been demonstrated that the non-integrable system the Kuramoto–Shivashinsky equation, does possess approximate Lie point and Lie–Bäcklund symmetries. In both the cases we have obtained the explicit form of the generators as a series in the small parameter ε. Lie point symmetry considerations lead to a similarity form which finally leads to a perturbed ordinary differential equation.

It is a well known fact that an integrable class of nonlinear equations in two spacetime dimensions does possess an infinite number of conserved quantities and hence an equal number of symmetry transformations [1]. On the other hand non-integrable systems do not share this beautiful property and show chaotic behaviour for some parameter values [2]. In a recent communication a slightly different approach for Lie point symmetry was presented by Baikov *et al* [3]. It was shown that one can generate non-trivial symmetries of the non-integrable systems if the terms for which the system is rendered non-integrable are treated as perturbation and an approximate form of Lie invariance is demanded. In the present work we have studied the famous Kuramoto–Shivashinsky (KS) [4] equation from this viewpoint and have deduced some non-trivial symmetry transformations for it. Earlier studies of the KS equation were from the point of view of chaos only. In the course of our analysis we have also deduced a similarity reduction of the KS equation which turns out to be an ordinary differential equation with some perturbation terms depending explicitly on the parameter t, the time variable.

For the sake of completeness, we describe here some aspects of approximate Lie symmetry. Let

$$y' = g(y, \varepsilon, a) \tag{1}$$

be a local transformation forming a one-parameter group with respect to the parameter a, so that

$$g(y, \varepsilon, 0) = y \qquad g\{g(y, \varepsilon, a), \varepsilon, b\} = g(y, \varepsilon, a+b). \tag{2}$$

Of course this type of composition law will also hold for transformations other than translation. We say $f \approx g$ if and only if

$$f(y, \varepsilon, a) = g(y, \varepsilon, a) + \mathrm{O}(\varepsilon^p) \tag{3}$$

where $\mathrm{O}(\varepsilon^p)$ denotes terms of the order of ε^p and the vector field corresponding to the above transformation is the tangent vector field

$$V(f) = \left.\frac{\partial f(y, \varepsilon, a)}{\partial a}\right|_{a=0} \tag{4}$$

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or for any smooth function $V(y, \varepsilon)$, the solution of the approximate Cauchy problem

$$\frac{\partial y'}{\partial a} \simeq V(y', \varepsilon) \qquad y'|_{a=0} \approx y \tag{5}$$

determines an approximate one-parameter group and was named as an approximate Lie equation by Baikov *et al* [3]. In the above discussion, ε is a small parameter, originally present in the system. It is of course possible to prove an approximate Lie theorem, with the above type of equivalence relation between functions.

After this brief introduction we may now state what is implied by approximate Lie invariance. The equation

$$F(y, \varepsilon) = 0 \tag{6}$$

is said to be approximately Lie invariant under the group of transformations $y' \approx f(y, \varepsilon, a)$ if

$$F(f(y, \varepsilon, a), \varepsilon) \approx 0 \tag{7}$$

and we can write the vector field as

$$X = (\xi_0(y, a) + \varepsilon \xi_1(y, a)) \frac{\partial}{\partial y} \tag{8}$$

or, in the general situation, in the form

$$X = (\xi_0(y, a) + \varepsilon \xi_1(y, a) + \varepsilon^2 \xi_2(y, a)) \frac{\partial}{\partial y}. \tag{9}$$

Let us now turn to the particular problem under consideration. The Kuramoto-Shivashinsky equation can be written as

$$U_t + uu_x + \varepsilon(\alpha u_{xx} + \gamma u_{xxxx}) = 0 \tag{10}$$

which actually governs the phenomenon of combustion, directional solidification and also weak two-dimensional turbulence. From the form of equation (10) we can interpret this as a perturbation of the $u_t + uu_x = 0$ problem, with ε being a small parameter.

We want to obtain a vector field X, which when operating on (10) will keep it invariant in the sense of equation (7). Let us denote the set of dependent and independent variables $\{u, x, t\}$ as Z_k and rewrite (10) as

$$F \equiv F_0(Z) + \varepsilon F_1(Z) = 0 \tag{11}$$

so we require

$$XF(Z, \varepsilon)_{F=0} = 0 \tag{12}$$

and we represent X as

$$X = (\xi_0^1 + \varepsilon \xi_1^1) \frac{\partial}{\partial t} + (\xi_0^2 + \varepsilon \xi_1^2) \frac{\partial}{\partial x} + (\eta_0 + \varepsilon \eta_1) \frac{\partial}{\partial u} = \Sigma(\xi_0^i + \varepsilon \xi_1^i) \frac{\partial}{\partial Z^i} \tag{13}$$

where $Z^1 = t_1$, $Z^2 = x$, $Z^3 = y = u$. Then it is easy to demonstrate that equation (10) leads to

$$\xi_0^K(y_0) \frac{\partial F_0(y_0)}{\partial Z^K} = 0 \tag{14}$$

$$\xi_1^K(y_0) \frac{\partial F_0(y_0)}{\partial Z^K} + \xi_0^K(y_0) \frac{\partial F_1(y_0)}{\partial Z^K} + y_1^l \frac{\partial}{\partial Z^l} \left(\xi_0^K(y_0) \frac{\partial F_0(y_0)}{\partial Z^K} \right) = 0 \tag{15}$$

where y_0 is the solution of the unperturbed equation. In the particular case under consideration we obtain from (14)

$$\begin{aligned} \eta_0 &= (-2g_1)u + h_3 \\ \xi_0^2 &= g_1 x + h_3 t + g_0 \\ \xi_0^1 &= 3g_1 t + f_0 \end{aligned} \tag{16}$$

where g_1, f_0, g_0, h_3 are arbitrary constants. In the next stage from equation (15) we get

$$\begin{aligned} \eta_1 &= -2l_1 u + K_3 \\ \xi_1^2 &= l_1 x + K_3 t + l_0 \\ \xi_1^1 &= 3l_1 t + e_0 \end{aligned} \tag{17}$$

along with $g_1 = 0$. So we have the following vector fields as the generator of approximate Lie transform:

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial X} \qquad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$$

$$X_4 = \varepsilon\left(3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}\right)$$

$$X_5 = \frac{\partial}{\partial t} \qquad X_6 = \varepsilon\left(t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right).$$

The Lie algebra generated by these vector fields is

$$\begin{aligned} &[X_1, X_3] = X_2 \qquad [X_1, X_4] = 3\varepsilon X_1 \qquad [X_1, X_6] = \varepsilon X_2 \\ &[X_2, X_4] = \varepsilon X_2 \qquad [X_4, X_3] = 2\varepsilon X_3 \qquad [X_3, X_5] = \varepsilon X_2 \\ &[X_4, X_5] = 3\varepsilon^2 X_1 \qquad [X_4, X_6] = 2\varepsilon^2 X_3 \\ &[X_5, X_6] = \varepsilon^2 X_2 \qquad \ldots . \end{aligned} \tag{18}$$

Once the symmetry generators are determined we can immediately obtain the similarity variables through:

$$\frac{\mathrm{d}t}{\xi_0^1 + \varepsilon\xi_1^1} = \frac{\mathrm{d}x}{\xi_0^2 + \varepsilon\xi_1^2} = \frac{\mathrm{d}u}{\eta_0 + \varepsilon\eta_1} \tag{19}$$

which upon integration immediately leads to the form:

$$u = q^{-2/3}\phi[(x - d_0 - b_0 q)q^{-1/3}] + b_0 \tag{20}$$

where ϕ is an arbitrary function and

$$q = t + a_0 \qquad a_0 = \frac{f_0 + \varepsilon e_0}{3\varepsilon l_1}$$

$$b_0 = \frac{h_3 + \varepsilon K_3}{2\varepsilon l_1} \qquad c_0 = \frac{g_0 + \varepsilon l_0}{3\varepsilon l_1}$$

$$S = (x - d_0 - b_0 q)q^{-1/3} \qquad d_0 = \tfrac{2}{3}b_0 a_0 - 3C_0.$$

Equation (20) when substituted in the Kuramoto–Sivashinsky equation (10) leads to

$$\phi\phi_s - \frac{S}{3}\phi_s - \tfrac{2}{3}\phi + \varepsilon(\alpha q^{1/3}\phi_{ss} + r q^{-1/2}\phi_{ssss}) = 0.$$

At this point we may note that the usual analysis of Lie point symmetry for the KS equation leads to a very simple trivial transformation law. It may be pointed out that with $g_1 = 0$, the expressions for $(\eta_0, \xi_0^2, \xi_0^1)$ are the same as that of Boling [6] but $(\eta_1, \xi_1^2, \xi_1^1)$ is a more general class of transformation in the next order. Furthermore it has been observed that Painlevé analysis yields some explicit solutions of these types of non-integrable nonlinear systems. Our conjecture is that a better idea about the singular manifold $\phi(xt) = 0$ can be obtained from the present analysis for such non-integrable systems.

Lastly we consider Lie-Bäcklund type symmetry for equations of this type. Let us consider first the equation,

$$u_t = -uu_x \tag{21a}$$

for which if we apply the criterion for the existence of a Lie-Bäcklund generator [6]

$$X(f^0) = f^0 \frac{\partial}{\partial u} + \text{necessary prolongation}$$

we get

$$f^0 = \phi(u)u_1 \tag{21b}$$

$\phi(u)$ being an arbitrary function of u, and $u_N = \partial^N u/\partial x^N$. We then consider the Kuramoto-Shivashinsky equation as a perturbation of (21a), and rewrite (10) as

$$u_t = -uu_x + \varepsilon H \tag{22}$$

with

$$H = -(au_4 + bu_2) \tag{23}$$

and write down the condition that the vector field

$$X(f) = f\frac{\partial}{\partial u}. \tag{24}$$

Here $f = \Sigma_0^p f^n \varepsilon^n$ is to be an approximate symmetry for (22). This leads to the following conditions of f^i:

$$\begin{aligned}\frac{\partial f^i}{\partial t} - h(u)\frac{\partial f^i}{\partial x} + \sum_{\alpha \geq 0} [D^\alpha(h(u)u_1) - h(u)u_{1+\alpha}]f^i_\alpha - h'(u)u_1 f^i \\ = + \sum_{\alpha \geq 0} [D^\alpha(f^{i-1})H_\alpha - f^{i-1}_\alpha D^\alpha(H)] \qquad i = 1, \ldots, p.\end{aligned} \tag{25}$$

Equation (25) gives a method for a recursive determination of the functions f^i.

From the term of the evolution equation we assume that

$$f^1 = e(u, u_1, u_2, u_3)u_4 + g(u, u_1, u_2, u_3) \tag{26}$$

and substitute in (25) when $i = 1$ and $f^0 = \phi(u)u_1$, whence from the coefficient of u_4 we get

$$u_1 \frac{\partial e}{\partial u_1} + 3u_2 \frac{\partial e}{\partial u_2} + \left(4u_3 + \frac{3u_2^2}{u_1}\right)\frac{\partial e}{\partial u_3} + 4e = 4a\phi'. \tag{27a}$$

Now set

$$e = a\frac{\partial\phi}{\partial u} - e'. \tag{27b}$$

Hence we immediately get

$$u_1\frac{\partial e'}{\partial u_1} + 3u_2\frac{\partial e'}{\partial u_2} + \left(4u_3 + \frac{3u_2^2}{u_1}\right)\frac{\partial e'}{\partial u_3} + 4e' = 0 \tag{28}$$

which can be solved and we get

$$e' = u_2^{-4/3}\theta\left[u_1u_2^{-1/3};\ u_3u_2^{-4/3} - \frac{3}{u_1}u_2^{2/3}\right] \tag{29}$$

where θ is an arbitrary function of the above two arguments written inside the bracket.

Now the remaining part of the equation (25), i.e.

$$\begin{aligned} u_1^2\frac{\partial g}{\partial u_1} &+ 3u_1u_2\frac{\partial g}{\partial u_2} + (4u_1u_3 + 3u_2^2)\frac{\partial g}{\partial u_3} + 10u_2u_3e - u_1g \\ &= 9a\phi''u_1u_2^2 + 10a\phi'u_2u_3 + 10a\phi''u_1^2u_3 + 10a\phi'''u_1^3u_2 + 6a\phi''u_1^2u_2 \\ &\quad + a\phi'u_1^4 + 2b\phi'u_1u_2 + b\phi''u_1^3 \end{aligned} \tag{30}$$

where e, in this equation, is to be substituted from (27b). An exact solution of (30) reads

$$\begin{aligned} g = \frac{10}{3}\left(\frac{u_3}{u_1}u_2^{-1/3} - \frac{u_2^{5/3}}{u_1^2}\right)\theta &+ \tfrac{3}{10}a\phi''u_2^2 + \tfrac{5}{2}a\phi''u_1u_3 + \tfrac{5}{2}a\phi'''u_1^2u_2 \\ &+ 2a\phi''\left[u_1u_2 + \frac{a}{2}\phi'u_1^3 + b\phi'u_2 + b\phi''u_1^2\right]. \end{aligned} \tag{31}$$

So the total solution is

$$f^1 = a\frac{\partial\phi}{\partial u}u_4 - u_2^{-4/3}\theta\left[u_1u_2^{-1/3},\ u_3u_2^{-4/3} - \frac{3}{u_1}u_2^{2/3}\right]u_4 + g \tag{32}$$

where g is given by (31) so the total vector is

$$X = (a\phi(u)u_1 + (\varepsilon f^1)\frac{\partial}{\partial u}$$

and the corresponding transformation is given as:

$$u \to u + \varepsilon a\phi(u)u_1 + \varepsilon^2 f^1 \tag{33}$$

keeping the equation invariant approximately up to terms of order $O(\varepsilon^2)$.

On the other hand we can also consider the Kuramoto–Shivashinsky equation as a perturbation of the well known Burger equation. So we may also consider:

$$u_t = (-uu_1 - bu_2) - \varepsilon au_4 \tag{34}$$

where b is not small. For the Lie point symmetry, proceeding as before we obtain the transformation

$$\begin{aligned} u &\to u + \varepsilon\eta_0 + \varepsilon^2\eta_1 \\ x &\to x + \varepsilon\xi_0^2 + \varepsilon^2\xi_1^2 \\ t &\to t + \varepsilon\xi_0^1 + \varepsilon^2\xi_1^1 \end{aligned} \tag{35}$$

where

$$\eta_0 = a_0' x + h_0' \qquad \xi_0^2 = h_0' t + b_0' \qquad \xi_0^1 = l_0$$

and

$$\begin{aligned} \eta_1 &= -(G_0 t + C_0) u + G_0 x + H_0 \\ \xi_1^2 &= (G_0 t + C_0) x + H_0 t + D_0 \\ \xi_1^1 &= G_0 t^2 + 2 C_0 t + E_0 \end{aligned}$$

where G_0, C_0, H_0, etc are arbitrary constants.

For the approximate Lie–Bäcklund symmetry, we keep (x, t) fixed but u is transformed as

$$u \to u + \varepsilon f^0 + \varepsilon^2 f^1$$

where $f^{0,1} = f^{0,1}(u, u_x, u_{xx}, \ldots)$.

The forms of f^0, f^1 are determined at stages. The calculation is very laborious so we omit the details and quote the result. For f^0 we get

$$f^0 = e_0' u_3 + \left(\frac{3e_0'}{2\alpha} u + g_0'\right) u_2 + \frac{3e_0'}{2\alpha} u_1^2 + \left(\frac{3e_0'}{4\alpha^2} u^2 - \frac{g_0'}{2\alpha} u + d_0'\right) u \tag{36}$$

$$f^1 = e_1(u, u_1, \ldots, u_5) u_6 + e_2(u, u_1, \ldots, u_5). \tag{37}$$

It turns out that e_1 is constant and e_2 has the following structure

$$\begin{aligned} e_2 = {} & (n_1 u_1 + n_2) u_3 + (n_1 u_2^2 + r_1 u_2 + r_2) u_5 + [\{-n_1 u_2 - u_1 (n_1' u_1 + n_2')\} u_3 \\ & + \{-4(n_1' \times u_1^2 + n_2' u_1 + \tfrac{1}{2} n_1' u_1 + \tfrac{1}{2} n_2') u_2^2 + K_5 u_2 + K_6\}] u_4 + \tfrac{1}{3} n_1 u_3^3 \\ & + (n_2' u_2 + 3 u_1 u_2 n_1' + n_1'' u_1^3 + n_2'' u_1^2) u_3^2 + K_3 u_3 + K_4 \end{aligned} \tag{38}$$

where

$$\begin{aligned} & n_1 = n_1(u) && n_2 = n_2(u) \\ & r_1 = r_1(u, u_1) && r_2 = r_2(u, u_1) \\ & K_5 = K_5(u, u_1) && K_6 = K_6(u, u_1) \\ & K_3 = K_3(u, u_1, u_2) && K_4 = K_4(u, u_1, u_2) \\ & K_{5,6} = K_{5,6}(u, u_1) \end{aligned}$$

are functions of their arguments shown above to be determined recursively. It is rather important to note that K_3, K_4 are actually determined in terms of the other functions. For example

$$\begin{aligned} K_3(u, u_1, u_2) = {} & \tfrac{7}{3} n_1' u_2^3 + \left\{\frac{15}{2\alpha} (u_1 n_1 + n_2) - u_1 (n_1'' u_1 + n_2'') - \frac{11}{2\alpha} n_1 u_1 \right. \\ & \left. + \tfrac{7}{2} n_1'' u_1^2 + 4 u_1^2 n_1'' + 4 u_1 n_2'' - \frac{\partial K_5}{\partial u_1}\right\} \frac{u_2^2}{2} \\ & + \left\{-\frac{e_0'^2}{2\alpha} + \frac{35 e_0'}{2\alpha} - \frac{4}{\alpha} u_1^3 n_1' - \frac{4}{\alpha} u_1^2 n_2' - n_1' \frac{u_1^3}{\alpha} - \frac{u_1^2}{2\alpha} n_2' \right. \\ & \left. - \tfrac{3}{2} u_1^4 n_1''' - \tfrac{3}{2} u_1^3 n_2''' - u_1 K_5' \right\} u_2 + K_7(u, u_1). \end{aligned} \tag{39}$$

In each of the above formulae n_i', K_i', etc denote the derivatives of the respective functions with respect to the variable u.

It is interesting to note that the form of the symmetry generator becomes completely different in the two cases as the nature of perturbing term is different.

So in the above analysis we have shown that even non-integrable systems may possess some approximate Lie point and Lie–Bäcklund symmetries. Such considerations may throw some light on the study of many more non-integrable equations which really outnumber the integrable ones.

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